XV. *A new Demonstration of the Binomial Theorem, when the Exponent is a positive or negative Fraction. By the Rev.* Abram Robertson, *A. M. F. R. S. Savilian Professor of Geometry in the University of Oxford. In a Letter to* Davies Giddy, *Esq. F. R. S.*

Read June 5, 1806.

DEAR SIR,

BEING perfectly convinced of your love of mathematical science, and your extensive acquirements in it, I submit to your perusal a new demonstration of the binomial theorem, when the exponent is a positive or negative fraction. As I am a strenuous advocate for smoothing the way to the acquisition of useful knowledge, I deem the following articles of some importance; and unless I were equally sincere in this persuasion, and in that of your desire to promote mathematical studies, in requesting the perusal, I should accuse myself of an attempt to trifle with your valuable time.

The following demonstration is new only to the extent above mentioned; but in order that the reader may perceive the proof to be complete, a successive perusal of all the articles is necessary. As far as it relates to the raising of integral powers, it is in substance the same with one which I drew up in the year 1794, and which was honoured with a place in the Philosophical Transactions for 1795. If, therefore,

you think the following demonstration worthy the attention of mathematicians, you will much oblige me by presenting it to the Royal Society.

I am, &c.

A. ROBERTSON.

Oxford,
March 21st, 1806.

1. The binomial theorem is a general expression for any power of the sum or difference of two quantities. Thus if n be any positive or negative whole number, or vulgar fraction, and a, b, be any two quantities, the binomial theorem expresses in a series the value of $\overline{a+b}|^n$, or $\overline{a-b}|^n$.

The binomial theorem is of very extensive utility. Besides the advantages derived from it in raising powers and extracting roots, it enables us to conduct, with clearness and ease, a variety of investigations in the higher parts of algebra, which, without its assistance, would become perplexed and laborious.

2. If n be a whole positive number, we can raise $x+a$ to the power denoted by n, in the following manner, by multiplication.

$$\begin{array}{l}
x+a \\
x+a \\
\hline
x^2+ax \\
\qquad ax+a^2 \\
\hline
x^2+2ax+a^2=\overline{x+a}|^2 \\
x+a \\
\hline
x^3+2ax^2+a^2x \\
\qquad ax^2+2a^2x+a^3 \\
\hline
x^3+3ax^2+3a^2x+a^3=\overline{x+a}|^3 \\
x+a \\
\hline
x^4+3ax^3+3a^2x^2+a^3x \\
\qquad ax^3+3a^2x^2+3a^3x+a^4 \\
\hline
x^4+4ax^3+6a^2x^2+4a^3x+a^4=\overline{x+a}|^4,
\end{array}$$

&c.

In the same manner the value of $\overline{x-a}|^n$ may be obtained; and its only difference from the value of $\overline{x+a}|^n$ will consist in having the negative sign prefixed to such terms as have an odd power of a. And as the powers of any other quantity, either simple or compound, may be obtained gradually by multiplying the last found power by the root, in order to find the next higher power, it is manifest that the principles of multiplication are the most simple and evident, to which we can resort, for the demonstration of the binomial theorem. These principles, therefore, will be used throughout the whole of the following investigations on the subject, and by them every case of the theorem will be established.

It is well known to mathematicians that the theorem has been repeatedly proved, either by induction, by the summation of figurate numbers, by the doctrine of combinations, by assumed series, or by fluxions; but that multiplication is a more direct way to the establishment of the theorem than any of these, cannot, I think, be doubted. Proceeding by multiplication, we have always an evident first principle in view, to which without the aid of any doctrine, foreign to the subject, we can appeal for the truth of our assertions, and the certainty and extent of our conclusions.

3. If p, q, be any two quantities, the product arising from the multiplication of p by q is equal to the product arising from the multiplication of q by p.* For magnitudes being to one another as their equimultiples, $p \times q : 1 \times q :: p : 1$, and $q \times p : 1 \times p :: q : 1$. But $1 \times q = q$, and $1 \times p = p$, and therefore, placing for ex æquali in a cross order,

$$p \times q : q : 1$$
$$q \times p : p : 1.$$

Consequently, $p \times q : 1 :: q \times p : 1$, and therefore $pq = qp$.

Hence it follows that the product arising from the multiplication of any number of quantities into one another, continues the same in value, in every variation which may be made in the arrangement of the quantities which compose it. Thus p, q, r, s, being any quantities, $pqrs = pqr \times s = spqr = spq \times r = rspq = rsp \times q = qrsp = qr \times s \times p = qr \times p \times s = qrps$, &c. And if $x + a = p$, $x + b = q$, $x + c = r$, $x + d = s$, $x + e = t$, &c. then $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} \times \overline{x+e} = pqrst = \overline{x+a} \times \overline{x+b} \times$

* When I speak of the multiplication of quantities into one another, I mean the multiplication of the numbers into one another which measure those quantities.

$\overline{x+c} \times \overline{x+e} \times \overline{x+d} = pqrts =$ any other arrangement which can take place in the quantities.

4. It is evident that each of the quantities a, b, c, &c. will be found the same number of times in the compound product arising from $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} \times \overline{x+e}$, &c. For this product is equal to $pqrst = pqrs \times \overline{x+e} = pqrt \times \overline{x+d} = pqst \times \overline{x+c} = prst \times \overline{x+b} = qrst \times \overline{x+a}$, by substituting for the compound quantities, $x+a$, $x+b$, &c. their equals p, q, &c. Wherefore, in the compound product, each of the quantities a, b, c, &c. will be found multiplied into the products of all the others.

5. These things being premised, we may poceed to the multiplication of the compound quantities $x+a$, $x+b$, $x+c$, &c. into one another; and in order to be as clear as possible in what follows, let us consider the sum of the quantities, a, b, c, &c. or the sum of any number of them multiplied into one another, as coefficients to the several powers of x, which arise in the multiplication. By considering products which contain the same number of the quantities a, b, c, &c. as homologous, the multiplication will appear as follows, and equations of various dimensions will arise, according to the powers of x.

$$\begin{array}{l} x+a=p \\ x+b=q \\ \hline \end{array}$$

$$x^2\left.\begin{array}{l}+a\\+b\end{array}\right\}x+ab=pq;$$ a quadratic equation, or an equation of two dimensions.

$$\begin{array}{l} x+c=r \\ \hline \end{array}$$

$$x^3\left.\begin{array}{l}+a\\+b\\+c\end{array}\right\}x^2\left.\begin{array}{l}+ab\\+ac\\+bc\end{array}\right\}x+abc=pqr;$$ a cubic, or an equation of three dimensions.

$$\begin{array}{l} x+d=s \\ \hline \end{array}$$

$$x^4\left.\begin{array}{l}+a\\+b\\+c\\+d\end{array}\right\}x^3\left.\begin{array}{l}+ab\\+ac\\+bc\\+ad\\+bd\\+cd\end{array}\right\}x^2\left.\begin{array}{l}+abc\\+abd\\+acd\\+bcd\end{array}\right\}x+abcd=pqrs;$$ a biquadratic, or an equation of four dimensions.

$$\begin{array}{l} x+e=t \\ \hline \end{array}$$

$$x^5\left.\begin{array}{l}+a\\+b\\+c\\+d\\+e\end{array}\right\}x^4\left.\begin{array}{l}+ab\\+ac\\+bc\\+ad\\+bd\\+cd\\+ae\\+be\\+ce\\+de\end{array}\right\}x^3\left.\begin{array}{l}+abc\\+abd\\+acd\\+bcd\\+abe\\+ace\\+bce\\+ade\\+bde\\+cde\end{array}\right\}x^2\left.\begin{array}{l}+abcd\\+abce\\+abde\\+acde\\+bcde\end{array}\right\}x+abcde=pqrst;$$ a sur-solid, or an equation of five dimensions.

&c.

6. From the above it appears, that the coefficient of the highest power of x in any equation is 1; but the coefficient of any other power of x in the same equation consists of a certain number of members, each of which contains one, two, three, &c. of the quantities a, b, c, &c. Thus the coefficient of

the second term of any equation is made up of members, each of which contains only one of the quantities a, b, c, &c. and the whole coefficient of the second term is the sum of all these members, or the sum of all the quantities a, b, c, &c. used in the multiplication by which the equation, under consideration, was produced. Thus in the equation of four dimensions, the whole coefficient of the second term is $a+b+c+d$, and a, b, c, d, were used in the multiplication in obtaining the equation. The coefficient of the third term, of any equation, is made up of members, each of which contains two of the quantities a, b, c, &c. used in the multiplication in obtaining the equation. Thus in the equation of four dimensions, the whole coefficient of the third term is $ab+ac+bc+ad+bd+cd$. And indeed, not only from inspection, but also from considering the manner in which the equations are generated, it is evident that each member of any coefficient has as many of the quantities a, b, c, &c. in it, as there are terms in the equation preceding the term to which the coefficient belongs. Thus each member of the coefficient in the second term of any equation is one quantity only, and only one term precedes the second term. Each member of the coefficient in the third term, of any equation, consists of two quantities, and two terms precede the third, &c.

7. When any equation is multiplied in order to produce the equation next above it, it is evident that the multiplication by x produces a part in the equation to be obtainea, which has the same coefficients as the equation multiplied. Thus, multiplying the equation of three dimensions by x we obtain that part of the equation of four dimensions which has the same

coefficients as the cubic: the only effect of this multiplication being the increase of the exponents of x by 1.

8. But when the same equation is multiplied by the quantity adjoined to x by the sign $+$, each term of the product, in order to rank under the same power of x, must be drawn one term back. Thus when the first term of the cubic is multiplied by d, the product must be placed in the second term of the biquadratic. When the second term of the cubic is multiplied by d, the product must be placed in the third term of the biquadratic: and so of others.

9. As the equation last produced is the product of all the compound quantities $x+a$, $x+b$, $x+c$, &c. into one another, and as it was proved in the fourth article that each of the quantities a, b, c, &c. must be found the same number of times in this product, if we can compute the number of times any one of those quantities enters into the coefficient of any term of the last equation, we shall then know how often each of the other enters into the same coefficient: and this may be done with ease, if of the quantities a, b, c, &c. we fix upon that used in the last multiplication. For the last equation, and indeed any other, may be considered as made up of two parts; the first part being the equation immediately before the last multiplied by x, according to the 7th article, and the second part being the same equation multiplied by the quantity adjoined to x by the sign $+$, last used in the multiplication, according to the 8th article. This last used quantity, therefore, never enters into the members of the coefficient of the first of these two parts, but it enters into all the members of the coefficients of the last of them. But that part into which it

does not enter has the same members as the coefficients of the equation immediately before the last, by the 7th article; and when the members of the first part are multiplied by the last used quantity, the product becomes the second part of the whole coefficient above mentioned.

Thus the first part of the cubic equation, by the 7th article is, $x^3 \begin{matrix} +a \\ +b \end{matrix}\Big\} x^2 + abx$, and as these coefficients are the same as the coefficients in the quadratic equation, being multiplied by c, and arranged according to the 8th article, we have the coefficients of the second part of the cubic, viz. $c \begin{matrix} +ac \\ +bc \end{matrix} + abc$.

Hence it is evident, that there are as many members in any coefficient, which have the last used quantity in them, as there are members in the coefficient preceding, which have not the same quantity. Thus in the 3d term, in the equation of four dimensions, there are three members of the whole coefficient of x^2 which have d in them, viz. ad, bd, cd, and there are three members of the whole coefficient of x^3 in the second term, which have not d in them, viz. a, b, c. In the fourth term of the same equation, there are three members of the whole coefficient of x, which have d in them, viz. abd, acd, bcd, and there are three members of the whole coefficient of x^2 in the third term which have not d in them, viz. ab, ac, bc. Now as it has been proved that each of the quantities a, b, c, &c. enters the same number of times into the coefficient of the same term, what has here been proved of the last used is applicable to each.

10. From the last article the number of members in the several coefficients of any equation may be determined. For

if we put $s=$ the number of times each quantity is found in a coefficient, $n=$ the number of quantities a, b, c, &c. used in producing the equation, and $p=$ the number of quantities in each member; then as a is found s times in this coefficient, b is found s times in this coefficient, &c. the number of quantities in this coefficient, with their repetitions, will be $s \times n$; and as p expresses the number of quantities requisite for each member, the number of members in the coefficient will be $\frac{sn}{p}$.

Thus, for the sake of illustration, if we limit the above notation to the second term of the equation of five dimensions, $s=1$, as each of the quantities a, b, c, &c. is found once in the whole coefficient of x^4; $p=1$, as each member consists of one quantity, and $n=5$, as a, b, c, d, e are used in producing the equation. Consequently $\frac{sn}{p}=5$. If we limit the above notation to the third term of the same equation, $s=4$, $p=2$, and $n=5$, and therefore $\frac{sn}{p}=10$. If we limit the above notation to the fourth term of the same equation, $s=6$, $p=3$, and $n=5$, and $\frac{sn}{p}=10$. If we limit the above notation to the fifth term of the same equation, $s=4$, $p=4$, and $n=5$, and $\frac{sn}{p}=5$.

11. Using the same notation, we can by the last two articles, calculate the number of members in the next coefficient after that whose number of members is $\frac{sn}{p}$. For as $\frac{sn}{p}$ expresses the number of members in the above mentioned coefficient, and s the number of times each quantity is found in it, $\frac{sn}{p}-s=$ the number of times each is not found in it. By the 9th article therefore, a will be found $\frac{sn}{p}-s$ times, b will be found $\frac{sn}{p}-s$ times, &c. in the next coefficient, and therefore $\overline{\frac{sn}{p}-s} \times n = \frac{sn^2-psn}{p} =$ the number of quantities, with

their repetitions, in it. But as the number of quantities in each member of a coefficient is 1 less than the number in each member of the coefficient next following, each member of the coefficient whose number of members we are now calculating will have in it $p+1$ number of quantities. Consequently $\frac{sn^2-psn}{p \times \overline{p+1}} = \frac{sn}{p} \times \frac{n-p}{p+1} =$ the number of members of the coefficient next after that whose number of members is $\frac{sn}{p}$, as in the last article.

12. It is evident, from the sixth article, that the value of p in the second term of any equation is 1; in the third term of any equation its value is 2; in the fourth term of any equation it is 3, &c. It is also evident that the number of members of the coefficient of the second term of any equation is n; for the whole coefficient is the sum of all the quantities a, b, c, &c. used in producing the equation. It therefore follows that the general expression $\frac{sn}{p} \times \frac{n-p}{p+1}$, obtained in the last article, enables us to ascertain the number of members in the coefficient of any term in an equation. For the number of members of the coefficient in the second term being n, according to the successive values of p the number of members in the third term is $n \cdot \frac{n-1}{2}$; in the fourth term it is $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$; in the fifth term it is $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$; and this regular form may be extended to express the number of members in the coefficient of any term whatever.

13. The binomial theorem, as far as it relates to the raising of integral powers, easily follows from the foregoing articles. For if all the quantities a, b, c, &c. used in the multiplication in the fifth article, be equal to one another, and consequently

each equal to a, each of the members in any coefficient will become a power of a; and, therefore, as the exponent of x in the first term is equal to n, it follows from the sixth and last articles that $\overline{x+a}|^n = x^n + nax^{n-1} + n \cdot \frac{n-1}{2} a^2 x^{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot a^3 x^{n-3} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^4 x^{n-4} +$ &c.

14. If equations be generated from $\overline{x-a} \cdot \overline{x-b} \cdot \overline{x-c} \cdot \overline{x-d}$, &c. the coefficients will be the same, excepting the signs, as those which result from $\overline{x+a} \cdot \overline{x+b} \cdot \overline{x+c} \cdot \overline{x+d}$, &c. in the fifth article; and as minus multiplied into minus gives plus, but minus multiplied into minus multiplied into minus gives minus, the coefficients in equations generated from $\overline{x-a} \cdot \overline{x-b} \cdot \overline{x-c} \cdot \overline{x-d}$, &c. whose members have each an even number of the quantities a, b, c, &c. will have the sign $+$, but coefficients whose members have each an odd number of the quantities a, b, c, &c. will have the sign $-$. And hence it is evident that $\overline{x-a}|^n = x^n - nax^{n-1} + n \cdot \frac{n-1}{2} a^2 x^{n-2} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^3 x^{n-3} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^4 x^{n-4} -$ &c.

15. By the general principles of involution $\overline{a+b}|^n = a^n \times \overline{1+\frac{b}{a}}|^n = a^n \times \overline{1+x}|^n$, by putting $x = \frac{b}{a}$. By article 13, $\overline{1+x}|^n = 1 + nx + n \cdot \frac{n-1}{2} x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} x^4 +$ &c. and by the same article $\overline{1+x}|^m = 1 + mx + m \cdot \frac{m-1}{2} x^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} x^3 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} x^4 +$ &c. But by the general principles of involution, and article 13, $\overline{1+x}|^n \times \overline{1+x}|^m = \overline{1+x}|^{n+m} = 1 + \overline{n+m}\, x + \overline{n+m} \cdot \frac{n+m-1}{2} x^2 + \overline{n+m} \cdot \frac{n+m-1}{2} \cdot \frac{n+m-2}{3} x^3 + \overline{n+m} \cdot \frac{n+m-1}{2} \cdot \frac{n+m-2}{3} \cdot \frac{n+m-3}{4} x^4 +$ &c. when n and m are whole numbers.

Hence it is evident that if the series equal to $\overline{1+x}|^n$ be multiplied by the series equal to $\overline{1+x}|^m$, the product must be equal to the series which is equal to $\overline{1+x}|^{n+m}$. Now the two first mentioned series being multiplied into one another, and the parts being arranged according to the powers of x, the several products will stand as in the following representation.

$$\overline{1+x}|^n = 1 + nx + n \cdot \frac{n-1}{2} x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} x^4 + \&c.$$

$$\overline{1+x}|^m = 1 + mx + m \cdot \frac{m-1}{2} x^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} x^3 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} x^4 + \&c.$$

$$\begin{array}{l} 1 + nx + n \cdot \frac{n-1}{2} x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} x^4 + \&c. \\ \quad mx + m \cdot nx^2 + m \cdot n \cdot \frac{n-1}{2} x^3 + m \cdot n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} x^4 + \&c. \\ \quad\quad m \cdot \frac{m-1}{2} x^2 + m \cdot \frac{m-1}{2} \cdot nx^3 + m \cdot \frac{m-1}{2} \cdot n \cdot \frac{n-1}{2} x^4 + \&c. \\ \quad\quad\quad m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} x^3 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot nx^4 + \&c. \end{array}$$

For the sake of reference hereafter let this be called multiplication A.

Now with respect to the coefficients prefixed to the several powers of x, in the foregoing multiplication, two observations are to be made, by means of which the demonstration of the theorem may be extended to fractional exponents.

In the first place, supposing n and m to be whole numbers, the sum of the coefficients prefixed to any individual power of x, in multiplication A, must be equal to the coefficient prefixed to the same power of x in the binomial series $1 + \overline{n+m}x + \overline{n+m} \cdot \frac{n+m-1}{2} x^2 + \overline{n+m} \cdot \frac{n+m-1}{2} \cdot \frac{n+m-2}{3} x^3 + \overline{n+m} \cdot \frac{n+m-1}{2} \cdot \frac{n+m-2}{3} \cdot \frac{n+m-3}{4} x^4 + \&c.$ The certainty of this circumstance rests partly on the 13th article, and partly on a

plain axiom, *viz.* that equals being multiplied by equals the products are equal.

In the second place it is to be observed, that the whole coefficient of any power of x, in the products of multiplication A, may be reduced to the regular binomial form, established in the 13th article. Thus $n \cdot \frac{n-1}{2} + mn + m \cdot \frac{m-1}{2}$, the whole coefficient of x^2, by actual multiplication becomes $\frac{n^2+m^2+2mn-n-m}{2} = \overline{n+m} \cdot \frac{n+m-1}{2}$. Also $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + mn \cdot \frac{n-1}{2} + m \cdot \frac{m-1}{2} \cdot n + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}$, the whole coefficient of x^3, by actual multiplication becomes $\frac{n^3+m^3-3n^2-3m^2+3n^2m}{6} + \frac{3m^2n-6mn+2n+2m}{6} = \overline{n+m} \cdot \frac{n+m-1}{2} \cdot \frac{n+m-2}{3}$. And from the preceding observation it is evident, that we may in the same manner, reduce the whole coefficient of any other power of x, in the products of multiplication A to the regular binomial form.

16. But in proceeding, as above, to change the form of the coefficients prefixed to any power of x, in multiplication A, into the regular binomial form, we are not under the necessity of supposing n and m to be whole numbers. The actual multiplications will end in the same powers of n and m, the same combinations of them, and the same numerals, whether we consider n and m as whole numbers or as fractions.

We are therefore at liberty to suppose n and m to be any two fractions whatever, in the two series multiplied into one another in multiplication A, and the same two fractions will take the place of n and m respectively in the regular binomial series $1 + \overline{n+m}\,x + \overline{n+m} \cdot \frac{n+m-1}{2} x^2 + \overline{n+m} \cdot \frac{n+m-1}{2} \cdot \frac{n+m-2}{3} x^3$

$+\overline{n+m}\,.\,\frac{n+m-1}{2}\,.\,\frac{n+m-2}{3}\,.\,\frac{n+m-3}{4}\,x^4+$ &c. which expresses the product of the two series into one another.

17. If therefore r be any positive whole number we can raise the binomial series $1+\frac{1}{r}x+\frac{1}{r}.\frac{\frac{1}{r}-1}{2}x^2+\frac{1}{r}.\frac{\frac{1}{r}-1}{2}.\frac{\frac{1}{r}-2}{3}x^3+\frac{1}{r}.\frac{\frac{1}{r}-1}{2}.\frac{\frac{1}{r}-2}{3}.\frac{\frac{1}{r}-3}{4}x^4+$ &c. to any proposed power by successive multiplications; or we can express any power of it by supposing the multiplications actually to have been gone through. Thus, calling the last mentioned series the root, if it be multiplied by itself, and if the coefficients in the product be expressed in the regular binomial form, its square will be $1+\frac{2}{r}x+\frac{2}{r}.\frac{\frac{2}{r}-1}{2}x^2+\frac{2}{r}.\frac{\frac{2}{r}-1}{2}.\frac{\frac{2}{r}-2}{3}x^3+\frac{2}{r}.\frac{\frac{2}{r}-1}{2}.\frac{\frac{2}{r}-2}{3}.\frac{\frac{2}{r}-3}{4}x^4+$ &c. Again, if this series be multiplied by the root, and the coefficients in the product be expressed in the regular binomial form, the cube of the root will be $1+\frac{3}{r}x+\frac{3}{r}.\frac{\frac{3}{r}-1}{2}x^2+\frac{3}{r}.\frac{\frac{3}{r}-1}{2}.\frac{\frac{3}{r}-2}{3}x^3+\frac{3}{r}.\frac{\frac{3}{r}-1}{2}.\frac{\frac{3}{r}-2}{3}.\frac{\frac{3}{r}-3}{4}x^4+$ &c. Proceeding thus, by multiplying the last found power by the root, in order to find the next higher power, the nth power of $1+\frac{1}{r}x+\frac{1}{r}.\frac{\frac{1}{r}-1}{2}x^2+\frac{1}{r}.\frac{\frac{1}{r}-1}{2}.\frac{\frac{1}{r}-2}{3}x^3+\frac{1}{r}.\frac{\frac{1}{r}-1}{2}.\frac{\frac{1}{r}-2}{3}.\frac{\frac{1}{r}-3}{4}x^4+$ &c. is $1+\frac{n}{r}x+\frac{n}{r}.\frac{\frac{n}{r}-1}{2}x^2+\frac{n}{r}.\frac{\frac{n}{r}-1}{2}.\frac{\frac{n}{r}-2}{3}x^3+\frac{n}{r}.\frac{\frac{n}{r}-1}{2}.\frac{\frac{n}{r}-2}{3}.\frac{\frac{n}{r}-3}{4}x^4+$ &c.

18. If in the series, which concludes the last article, n be equal to r, the whole series becomes equal to $1+x$. For in this case $\frac{n}{r}=1$, and therefore $\frac{\frac{n}{r}-1}{2}=0$, and consequently every term in the series, after the second, becomes equal to 0, or vanishes.

Hence it is evident that the rth root of $1+x$, or, which is the same thing, that $\overline{1+x}|^{\frac{1}{r}}=1+\frac{1}{r}x+\frac{1}{r}\cdot\frac{\frac{1}{r}-1}{2}x^2+\frac{1}{r}\cdot\frac{\frac{1}{r}-1}{2}\cdot\frac{\frac{1}{r}-2}{3}x^3+\frac{1}{r}\cdot\frac{\frac{1}{r}-1}{2}\cdot\frac{\frac{1}{r}-2}{3}\cdot\frac{\frac{1}{r}-3}{4}x^4+$ &c. for this series being raised to the rth power becomes equal to $1+x$.

As by the general principles of involution the nth power of $\overline{1+x}|^{\frac{1}{r}}$ is $\overline{1+x}|^{\frac{n}{r}}$, it therefore follows, from the last observation and the preceding article, that $\overline{1+x}|^{\frac{n}{r}}=1+\frac{n}{r}x+\frac{n}{r}\cdot\frac{\frac{n}{r}-1}{2}x^2+\frac{n}{r}\cdot\frac{\frac{n}{r}-1}{2}\cdot\frac{\frac{n}{r}-2}{3}x^3+\frac{n}{r}\cdot\frac{\frac{n}{r}-1}{2}\cdot\frac{\frac{n}{r}-2}{3}\cdot\frac{\frac{n}{r}-3}{4}x^4+$ &c.

19. By the general principles of involution $\overline{a-b}|^n=a^n\times\overline{1-\frac{b}{a}}|^n=a^n\times\overline{1-x}|^n$, by putting $x=\frac{b}{a}$. By article 14, n being a whole number, $\overline{1-x}|^n=1-nx+n\cdot\frac{n-1}{2}x^2-n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^3+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}x^4-$ &c. and by the same article, m being a whole number, $\overline{1-x}|^m=1-mx+m\cdot\frac{m-1}{2}x^2-m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}x^3+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}x^4-$ &c. But by the general principles of involution, and article 14, $\overline{1-x}|^n\times\overline{1-x}|^m=\overline{1-x}|^{n+m}$

$$=1-\overline{n+m}x+\overline{n+m}\cdot\frac{n+m-1}{2}x^2-\overline{n+m}\cdot\frac{n+m-1}{2}\cdot\frac{n+m-2}{3}x^3$$
$$+\overline{n+m}\cdot\frac{n+m-1}{2}\cdot\frac{n+m-2}{3}\cdot\frac{n+m-3}{4}x^4-\text{&c.}$$

Hence it is evident that if the series equal to $\overline{1-x}|^n$ be multiplied by the series equal to $\overline{1-x}|^m$, the product must be equal to the series, which is equal to $\overline{1-x}|^{n+m}$. Now the two first mentioned series being multiplied into one another, and the parts being arranged according to the powers of x, the several products will stand as in the following representation.

$$\overline{1-x}|^n=1-nx+n\cdot\frac{n-1}{2}x^2-n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^3+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}x^4-\text{&c.}$$
$$\overline{1-x}|^m=1-mx+m\cdot\frac{m-1}{2}x^2-m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}x^3+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}x^4-\text{&c.}$$

$$\begin{array}{llllll}
1 & -nx & +n\cdot\frac{n-1}{2}x^2 & -n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^3 & +n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}x^4 & -\text{&c.}\\
 & -mx & +m\cdot nx^2 & -m\cdot n\cdot\frac{n-1}{2}x^3 & +m\cdot n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^4 & -\text{&c.}\\
 & & m\cdot\frac{m-1}{2}x^2 & -m\cdot\frac{m-1}{2}\cdot nx^3 & +m\cdot\frac{m-1}{2}\cdot n\cdot\frac{n-1}{2}x^4 & -\text{&c.}\\
 & & & -m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}x^3 & +m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot nx^4 & -\text{&c.}
\end{array}$$

For the sake of reference hereafter let this be called multiplication B.

Now for the same reasons as are stated in the 15th and 16th articles, the whole coefficient prefixed to any power of x in multiplication B, must be equal to the coefficient prefixed to the same power of x in the series $1-\overline{m+n}x+\overline{m+n}\cdot\frac{m+n-1}{2}x^2-\overline{m+n}\cdot\frac{m+n-1}{2}\cdot\frac{m+n-2}{3}x^3+\overline{m+n}\cdot\frac{m+n-1}{2}\cdot\frac{m+n-2}{2}\cdot\frac{m+n-3}{4}x^4-$ &c.; and we are also at liberty to suppose n and m to be any two fractions whatever, in the series multiplied into one another, and consequently in the series expressing their product.

Proceeding therefore as in the 17th and 18th articles, and using the same notation, $\overline{1-x}|^{\frac{1}{r}} = 1 - \frac{1}{r}x + \frac{1}{r} \cdot \frac{\frac{1}{r}-1}{2}x^2 - \frac{1}{r} \cdot \frac{\frac{1}{r}-1}{2} \cdot \frac{\frac{1}{r}-2}{3}x^3 + \frac{1}{r} \cdot \frac{\frac{1}{r}-1}{2} \cdot \frac{\frac{1}{r}-2}{3} \cdot \frac{\frac{1}{r}-3}{4}x^4 - \text{\&c.}$

Also $\overline{1-x}|^{\frac{n}{r}} = 1 - \frac{n}{r}x + \frac{n}{r} \cdot \frac{\frac{n}{r}-1}{2}x^2 - \frac{n}{r} \cdot \frac{\frac{n}{r}-1}{2} \cdot \frac{\frac{n}{r}-2}{3}x^3 + \frac{n}{r} \cdot \frac{\frac{n}{r}-1}{2} \cdot \frac{\frac{n}{r}-2}{3} \cdot \frac{\frac{n}{r}-3}{4}x^4 - \text{\&c.}$

20. It is easily proved, by means of the 15th and 16th articles, that

$$\frac{\overline{1+x}|^m}{\overline{1+x}|^n} \text{ or } \frac{1 + mx + m \cdot \frac{m-1}{2}x^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}x^3 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4}x^4 + \text{\&c.}}{1 + nx + n \cdot \frac{n-1}{2}x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}x^4 + \text{\&c.}}$$

is equal to the series $1 + \overline{m-n}x + \overline{m-n} \cdot \frac{m-n-1}{2}x^2 + \overline{m-n} \cdot \frac{m-n-1}{2} \cdot \frac{m-n-2}{3}x^3 + \overline{m-n} \cdot \frac{m-n-1}{2} \cdot \frac{m-n-2}{3} \cdot \frac{m-n-3}{4}x^4 + \text{\&c.}$ whether m and n be whole numbers or fractions. For v being equal to $m-n$, this last series becomes $1 + vx + v \cdot \frac{v-1}{2}x^2 + v \cdot \frac{v-1}{2} \cdot \frac{v-2}{3}x^3 + v \cdot \frac{v-1}{2} \cdot \frac{v-2}{3} \cdot \frac{v-3}{4}x^4 + \text{\&c.}$; and this series being multiplied by $1 + nx + n \cdot \frac{n-1}{2}x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-1}{3} \cdot \frac{n-3}{4}x^4 + \text{\&c.}$ the series expressing their product, by the 15th and 16th articles, is $1 + \overline{v+n}x + \overline{v+n} \cdot \frac{v+n-1}{2}x^2 + \overline{v+n} \cdot \frac{v+n-1}{2} \cdot \frac{v+n-1}{3}x^3 + \overline{v+n} \cdot \frac{v+n-1}{2} \cdot \frac{v+n-2}{3} \cdot \frac{v+n-3}{4}x^4 + \text{\&c.}$ But as v is equal to $m-n$, this last series is equal to $1 + mx + m \cdot \frac{m-1}{2}x^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}x^3 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4}x^4 + \text{\&c.}$

Hence it is evident that $\frac{\overline{1+x}|^m}{\overline{1+x}|^n}$ is equal to $1 + \overline{m-n}x + \overline{m-n}$

$\frac{m-n-1}{2}x^2+\overline{m-n}\cdot\frac{m-n-1}{2}\cdot\frac{m-n-2}{3}x^3+\overline{m-n}\cdot\frac{m-n-1}{2}\cdot\frac{m-n-2}{3}\cdot\frac{m-n-3}{4}x^4+$ &c.; and as this equation holds in every possible value of m, and as, by the general principles of involution $\overline{1+x}|^0$ is equal to 1, when m is equal to o then $\frac{1}{\overline{1+x}|^n}$, or $\overline{1+x}|^{-n}=1-nx-n\cdot\frac{-n-1}{2}x^2-n\cdot\frac{-n-1}{2}\cdot\frac{-n-2}{3}x^3-n\cdot\frac{-n-1}{2}\cdot\frac{-n-2}{3}\cdot\frac{-n-3}{4}x^4-$ &c.

According to the form of the binomial series, the whole of the second, fourth, sixth, &c. terms in the last series consist of an odd number of negative parts multiplied into one another, and therefore each of these terms becomes a negative quantity. But the whole of the third, fifth, seventh, &c. terms, consist of an even number of negative parts multiplied into one another, and therefore each of these terms becomes a positive quantity. Consequently, $\overline{1+x}|^{-n}=1-nx+n\cdot\frac{n+1}{2}x^2-n\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}x^3+n\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}\cdot\frac{n+3}{4}x^4-$ &c.

21. By the 19th article we are enabled to prove that

$$\frac{\overline{1-x}|^m}{\overline{1-x}|^n}\text{ or }\frac{1+m\cdot-x+m\cdot\frac{m-1}{2}x^2+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot-x^3+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}x^4+\&c.}{1+n\cdot-x+n\cdot\frac{n-1}{2}x^2+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot-x^3+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}x^4+\&c.}$$

is equal to the series $1+\overline{m-n}\cdot-x+\overline{m-n}\cdot\frac{m-n-1}{2}x^2+\overline{m-n}\cdot\frac{m-n-1}{2}\cdot\frac{m-n-2}{3}\cdot-x^3+\overline{m-n}\cdot\frac{m-n-1}{2}\cdot\frac{m-n-2}{3}\cdot\frac{m-n-3}{4}x^4+$ &c. For, as in the preceding article, if this last series be multiplied by $1+n\cdot-x+n\cdot\frac{n-1}{2}x^2+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot-x^3+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}x^4+$ &c. the series expressing the product will be $1+m\cdot-x+m\cdot\frac{m-1}{2}x^2+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot-x^3+m\cdot\frac{m-1}{2}\cdot$

$\frac{m-2}{3} \cdot \frac{m-3}{4} x^4 +$ &c. Consequently as $\frac{\overline{1-x}|^m}{\overline{1-x}|^n} = 1 + \overline{m-n} \cdot -x + \overline{m-n} \cdot \frac{m-n-1}{2} x^2 + \overline{m-n} \cdot \frac{m-n-1}{2} \cdot \frac{m-n-2}{3} \cdot -x^3 + \overline{m-n} \cdot \frac{m-n-1}{2} \cdot \frac{m-n-2}{3} \cdot \frac{m-n-3}{4} x^4 +$ &c. in every possible value of m, it follows that when m is equal to 0, then $\frac{1}{\overline{1-x}|^n}$ or $\overline{1-x}|^{-n}$ $= 1 - n \cdot -x - n \cdot \frac{-n-1}{2} x^2 - n \cdot \frac{-n-1}{2} \cdot \frac{-n-2}{3} \cdot -x^3 - n \cdot \frac{-n-1}{2} \cdot \frac{-n-2}{3} \cdot \frac{-n-3}{4} x^4 -$ &c.

The form of this series, however, may be changed into one more convenient. For the whole of the second, fourth, sixth, &c. terms consist of an even number of negative parts multiplied into one another, and therefore each of these terms becomes a positive quantity. And as the coefficients of the third, fifth, seventh, &c. terms consist of an even number of negative parts multiplied into one another, and as in these terms the powers of x are positive, each of these terms becomes a positive quantity. Consequently $\overline{1-x}|^{-n} = 1 + nx + n \cdot \frac{n+1}{2} x^2 + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} x^3 + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} x^4 +$ &c.

Every particular necessary for the establishment of the binomial theorem has now been proved. I therefore proceed to conclude the subject, by shewing that each of the four forms, in which the theorem may be expressed, immediately follows from the preceding articles, and the general principles of involution. In each of them n is to be considered either as a whole number or fraction.

22. By article 18, $\overline{1+x}|^n = 1 + nx + n \cdot \frac{n-1}{2} x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} x^4 +$ &c. But if $\frac{b}{a}$ be equal to x, then

$a^n \times \overline{1+\frac{b}{a}}|^n = \overline{a+b}|^n$, by the general principles of involution; and therefore $\overline{a+b}|^n = a^n \times : 1 + n\frac{b}{a} + n \cdot \frac{n-1}{2}\frac{b^2}{a^2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{b^3}{a^3} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}\frac{b^4}{a^4} +$ &c. $= a^n + nba^{n-1} + n \cdot \frac{n-1}{2} b^2 a^{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} b^3 a^{n-3} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} b^4 a^{n-4} +$ &c.

By article 19, $\overline{1-x}|^n = 1 - nx + n \cdot \frac{n-1}{2} x^2 - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} x^4 -$ &c. and therefore as before, if $\frac{b}{a}$ be equal to x, $\overline{a-b}|^n = a^n - nba^{n-1} + n \cdot \frac{n-1}{2} ba^{n-2} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} b^3 a^{n-3} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} b^4 a^{n-4} -$ &c.

By article 20, $\overline{1+x}|^{-n} = 1 - nx + n \cdot \frac{n+1}{2} x^2 - n \cdot \frac{n+1}{2} \cdot \frac{n-2}{3} x^3 + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} x^4 -$ &c. and therefore if $\frac{b}{a}$ be equal to x, $\overline{1+\frac{b}{a}}|^{-n} = 1 - n\frac{b}{a} + n \cdot \frac{n+1}{2}\frac{b^2}{a^2} - n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}\frac{b^3}{a^3} + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4}\frac{b^4}{a^4} -$ &c. But by the general principles of involution $\frac{1}{a^n \times \overline{1+\frac{b}{a}}|^n} = a^{-n} \times \overline{1+\frac{b}{a}}|^{-n} = \overline{a+b}|^{-n}$; and therefore $\overline{a+b}|^{-n} = a^{-n} - nba^{-n-1} + n \cdot \frac{n+1}{2} b^2 a^{-n-2} - n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} b^3 a^{-n-3} + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} b^4 a^{-n-4} -$ &c.

By article 21, $\overline{1-x}|^{-n} = 1 + nx + n \cdot \frac{n+1}{2} x^2 + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} x^3 + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} x^4 +$ &c.; and therefore if $\frac{b}{a}$ be equal to x, $\overline{1-\frac{b}{a}}|^{-n} = 1 + n\frac{b}{a} + n \cdot \frac{n+1}{2}\frac{b^2}{a^2} + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}\frac{b^3}{a^3} + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4}\frac{b^4}{a^4} +$ &c. But by the general principles of involution $\frac{1}{a^n \times \overline{1-\frac{b}{a}}|^n} = a^{-n} \times \overline{1-\frac{b}{a}}|^{-n} = \overline{a-b}|^{-n}$; and therefore $\overline{a-b}|^{-n} =$

$a^{-n} + nba^{-n-1} + n \cdot \frac{n+1}{2} b^2 a^{-n-2} + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} b^3 a^{-n-3} + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} b^4 a^{-n-4} +$ &c.

The four forms expressed in this article include the whole of the binomial theorem.